South East Asian J. of Math. & Math. Sci. **ISSN : 0972-7752**
Vol.14, No.2 (2018), pp. 45 - 50

Some Results in Fixed Point Theorems for Asymptotically Regular Mappings in *G*-Matric Spaces

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Abstract : In this paper we extend some results of fixed points for asymptotically regular mappings on a complete G-metric spaces by using new approach.

Keywords and phrases : Asymptotically Regular Mapping, Fixed Point, G-Cauchy sequence, G –Metric spaces.

1. Intoduction

Banach fixed point theorem is an important tool in the theory of metric spaces, it guarantees the existence and uniqueness of fixed points of self maps of metric spaces. The concept of asymptotically regular at a point in a space was first introduced by Browder and Petryshyn. In this paper we have proved some results in Complete G-metric space under asymptotic regularity with some contractive condition.

2 Definition and Preliminaries :

Definition : 2.1

Let X be a non empty set, and let $G : X \times X \times X \to R^+$ be a function satisfying the following properties.

(1) $G(x, y, z) = 0$ if $x = y = z$.

(2) $G(x, x, y) > 0$; for all $x, y \in X$ with $x \neq y$.

(3) $G(x, x, y) \leq G(x, y, z)$ for all x, y, z with $y \neq z$

(4) $G(x, y, z) = G(x, z, y) = G(y, x, z) = \ldots$ (Symmetry in all three variables)

(5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality)

Then the function G is called a generalized metric or more specifically a G-metric on X and the pair (X, G) is called a G-metric space.

Definition : 2.2

Let (X, G) be a G-metric space and let $\{x_n\}$ be a sequence of points of X. A point x in X is said to be the limit of the sequence $\{x_n\}$ if $\lim_{n\to\infty} G(x_n, x_n\ x) = 0$ and one say that the sequence $\{x_n\}$ is G-convergent to x.

Definition : 2.3

Let (X, G) be a G-metric space and let $\{x_n\}$ is called G-Cauchy if given $\epsilon > 0$, there is $N \in \mathbb{N}$ such that $G(x_n, x_m, x_\ell) < \epsilon$, for all n, m, $\ell \geq N$.

i.e $G(x_n, x_m, x_\ell) \to 0$, as n, m, $\ell \to \infty$.

Definition : 2.4

A mapping $T : X \to X$ of a symmetric G-metric space (X, G) into itself is said to be asymptotically regular at a point $x \in X$,

$$\text{if} \quad \lim_{n\to\infty} G(T^{n+1}x, T^n x, T^n x) = \lim_{n\to\infty} G(T^n x, T^{n+1}x, T^{n+1}x) = 0.$$

Where $T^n x$ is n^{th} iterate of T at $x \in X$

Proposition : 2.5

Every G-metric space (X, G) induces a metric space (X, d_G)

$$\text{where } d_G(x, y) = G(x, y, y) + G(y, x, x),\ \forall\ x, y \in X \tag{1}$$

Proof.

(i) $d_G(x, y) = G(x, y, y) + G(y, x, x) \geq G(x, x, x) = 0$ (by rectangular inequality)

(ii) $d_G(x, x) = G(x, x, x) + G(x, x, x) = 0$. (i.e $d_G(x, y) = 0$ if and only if $x = y$.)

(iii) $d_G(x, y) = G(x, y, y) + G(y, x, x) = G(y, x, x) + G(x, y, y) = d_G(y, x)$

(iv) $d_G(x, y) = G(x, y, y) + G(y, x, x)$

$$\begin{aligned} &\leq G(x, z, z) + G(z, y, y) + G(y, z, z) + G(z, x, x) \\ &\leq [G(x, z, z) + G(z, x, x)] + [G(z, y, y) + G(y, z, z)] \\ &\leq d_G(x, z) + d_G(z, y) \end{aligned}$$

Hence (X, G) induces a metric space (X, d_G)

3 The Main Result

Theorem : 3.1

Let (X, G) be a complete G-metric space and $T: X \to X$ be a Self map which is continuous such that the following condition is satisfied

$$G(Tx, Ty, Ty) \leq aG(x, y, y) + b[G(x, Tx, Tx) + G(Ty, y, y)] + c[G(x, Ty, Ty) + G(Tx, y, y)] \tag{2}$$

$$G(Ty, Tx, Tx) \leq aG(y, x, x) + b[G(y, Ty, Ty) + G(Tx, x, x)] + c[G(y, Tx, Tx) + G(Ty, x\ x] \tag{3}$$

for all x, y in X. where $0 \leq a + 2c, b + 2c < 1$, then T has a unique fixed point in X, if T is asymptotically regular at some point in X.

Proof.

We shall assume that T is asymptotically regular at a point $x_0 \in X$. and consider the sequence $\{T^n x_0\}$.

Adding (2) and (3) and applying (1) we get

$$d_G(Tx, Ty) \leq ad_G(x, y) + b[d_G(x, Tx) + d_G(y, Ty)] + c[d_G(x, Ty) + d_G(y, Tx)] \tag{4}$$

now

$$\begin{aligned} & d_G(T^n x_0, T^m x_0) \\ & \leq ad_G(T^{n-1}x_0, T^{m-1}x_0) + b[d_G(T^{n-1}x_0, T^n x_0) + d_G(T^{m-1}x_0, T^m x_0)] \\ & \qquad + c[d_G(T^{n-1}x_0, T^m x_0) + d_G(T^{m-1}x_0, T^n x_0)] \\ & \leq a[d_G(T^{n-1}x_0, T^n x_0) + d_G(T^n x_0, T^m x_0) + d_G(T^m x_0, T^{m-1}x_0)] \\ & \qquad + b[d_G(T^{n-1}x_0, T^n x_0) + d_G(T^{m-1}x_0, T^m x_0)] \\ & \qquad + c[d_G(T^{n-1}x_0, T^n x_0) + d_G(T^n x_0, T^m x_0) + d_G(T^{m-1}x_0, T^m x_0) + d_G(T^m x_0, T^n x_0)] \end{aligned}$$

i.e., $(1 - a - 2c)\, d_G(T^n x_0, T^m x_0) \leq (a + b + d)\, [d_G(T^{n-1}x_0, T^n x_0) + d_G(T^{m-1}x_0, T^m x_0)]$

$$d_G(T^n x_0, T^m x_0) \leq \frac{(a+b+d)}{(1-a-2c)} [d_G(T^{n-1}x_0, T^n x_0) + d_G(T^{m-1}x_0, T^m x_0)]$$

Since T is asymptotically regular at x_0,

$$d_G(T^n x_0, T^m x_0) \to 0 \text{ as } m, n \to \infty$$

because $d_G(T^{n-1}x_0, T^n x_0) \to 0$

$$d_G(T^{m-1}x_0, T^m x_0) \to 0 \text{ as } m, n \to \infty$$

Hence $\{T^n x_0\}$ is a Cauchy sequence.

Since (X,G) is complete, there exist a point $u \in X$ such that $u = \lim_{n \to \infty} T^n x_0$

Suppose that u is not a fixed point of T,

$$
\begin{aligned}
d_G(u, Tu) &= d_G(T^n x_0, Tu) \\
&\le a d_G(T^{n-1}x_0, u) + b\,[d_G(T^{n-1}x_0, T^n x_0) + d_G(u, Tu)] \\
&\qquad + c[d_G(T^{n-1}x_0, Tu) + d_G(u, T^n x_0)]
\end{aligned}
$$

Taking the limit as $n \to \infty$, we obtain

$d_G(u, Tu) \le (b + 2c)\, d_G(u, Tu)$

Which contradicts $(b + 2c) < 1$ unless $u = Tu$,

Suppose T has second fixed point v in X

$$
\begin{aligned}
d_G(u, v) &= d_G(Tu, Tv) \\
&\le a\, d_G(u, v) + b\,[d_G(u, Tu) + d_G(v, Tv)] + c[d_G(u, Tv) + d_G(v, Tu)] \\
&\le (a + 2c)\, d_G(u, v)
\end{aligned}
$$

Which contradicts $(a + 2c) < 1$ unless $u = v$.

Hence the fixed point is unique.

Theorem : 3.2

Let (X, G) be a complete G-metric space and $T: X\to X$ be a Self map which is continuous such that for every $x \in X$, $\{T^n x\}$ is a Cauchy sequence. Then T has a fixed point.Further if there is an injective self map $S : X \to X$ with $\alpha \in (0, 1)$ and $\beta < 1/2$ and

$G(STx, STy, STy)$

$$\le \alpha[G(Sx, STx, STx) + G(STy, Sy, Sy)] + \beta[G(Sx, STy, STy) + G(STx, Sy, Sy)] \tag{5}$$

$G(STy, STx, STx)$

$$\le \alpha[G(Sy, STy, STy) + G(STx, Sx, Sx)] + \beta[G(Sy, STx, STx) + G(STy, Sx,Sx] \tag{6}$$

for all $x, y \in X$, then the fixed point is unique and for any $x_0 \in X$, the sequence $\{T^n x_0\}$ converges to the unique fixed point.

Proof.

Let $x_0 \in X$ and define the sequence $x_{n+1} = T(x_n)$ for $n = 0, 1, 2,\ldots$ Now, since $\{T^n x_0\}$ is Cauchy sequence and X is complete, there exist $x \in X$ such that $T^n x_0 \to x$. Again since $T^n x_0 = x_n$, we have that $x_n \to x$. By continuity of T, we have $Tx_n \to Tx$. Now, since the sequences x_n and Tx_n are essentially same, thus we have $Tx = x$.

Adding (5) and (6) and applying (1) we get

$$d_G(STx, STy) \le \alpha[d_G(Sx, STx) + d_G(Sy, STy)] + \beta[d_G(Sx, STy) + d_G(Sy, STx)] \tag{7}$$

let u, v $\in$ X be fixed points of T, i.e., Tx = x and Ty = y , then

$$d_G(Sx, Sy) = d_G(STx, STy)$$
$$\leq \alpha[d_G(Sx, STx) + d_G(Sy, STy)] + \beta[d_G(Sx, STy) + d_G(Sy, STx)]$$
$$= \alpha[d_G(Sx, Sx) + d_G(Sy, Sy)] + \beta[d_G(Sx, Sy) + d_G(Sy, Sx)]$$
$$= 2\beta d_G(Sx, Sy).$$

Which contradicts $\beta < 1/2$ unless Sx = Sy. Now S being injective, we have x = y.

Hence for each $x_0 \in X$, the sequence $\{T^n x_0\}$ converges to the unique fixed point.

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